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NOTE

A Note on Log Scale Hankel Transforms

The Siegman–Talman approximation of the Hankel transform may suffer from rounding errors caused by premultiplication by an exponential function. The main result of this note is a modification of the Siegman–Talman algorithm for the approximation of the Hankel transform H_0. The modified algorithm is just as fast as the Siegman–Talman algorithm, but it cures the problem of rounding errors.

1. INTRODUCTION

The Hankel transform plays a part in various branches of physics: acoustics, optics, geo-electricity. This note grew out of some work in geo-hydrology. For a perfectly layered earth we consider the determination of the piezometric head or potential ϕ. At the surface the piezometric head should vanish, and in one of the layers there is a well, modeled as a Dirac delta function. Making use of the continuity equation and Darcy's law, one easily sees that the mathematical formulation of this problem is very much like the mathematical formulation of the geo-electrical problem as considered by Koefoed [5]. We refer to his monograph for a derivation of the results. Koefoed describes the potential ϕ_j in the jth layer as a perturbation of the potential for a Dirac delta function in a homogeneous earth. The perturbation is in the form of a Hankel transform (cylinder coordinates)

$$\int_0^\infty K_j(\lambda)\, J_\nu(\lambda r)\, d\lambda, \qquad \nu = 0, 1, \tag{1.1}$$

where K_j is obtained by a recurrence relation, see Koefoed [5]. For stability aspects of this recurrence relation see [11].

The approximate computation of the transform (1.1) involves some form of discretization. In some areas it is not advantageous to sample the function K_j with exponential spacing, see Candel [2], Piessens and Branders [8], and Wong [12] for methods which avoid exponential spacing. However, in other areas there is no problem with an exponential spacing, see Talman [10] for some examples from physics. In Koefoed's work, too, the exponential spacing is no problem, see Gosh [4]. This means that the Gardner transform is applicable, see Gardner [3], resulting in the Siegman–Talman approximation method for (1.1), cf. Siegman [9], Talman [10]. We shall explain this method in detail in Section 2.

Some experiments with the Siegman–Talman approximation method showed bad results either for large values of r, or for small values of r. The errors turned out to be rounding errors that were blown up by an exponential function e^{ar}. For the Hankel transform H_ν, $\nu > 0$ this exponential function is easily avoided. We describe this in detail in the preliminaries, Section 2. The main result of this note is an algorithm for the Hankel transform H_0 without exponential blowup of rounding errors, see Section 3. The algorithm is a slight modification of the Siegman–Talman algorithm. Finally, we give a numerical example in Section 4.

2. PRELIMINARIES: THE HANKEL TRANSFORMATION H_ν, $\nu > 0$

In this section we discuss the approximation of the Hankel transform corresponding to J_ν by means of a discrete Fourier transform. The Hankel transform H_ν is defined as

$$H_\nu f(r) = \int_0^\infty f(\lambda)\, J_\nu(\lambda r)\, d\lambda, \qquad r > 0. \tag{2.1}$$

Some authors employ a different but related definition. Following Gardner *et al.* [3] and Gosh [4] we substitute

$$r = e^x, \qquad \lambda = e^{-y}, \tag{2.2}$$

resulting in

$$(H_\nu f)(e^x) = \int_{-\infty}^\infty e^{-y} f(e^{-y})\, J_\nu(e^{x-y})\, dy. \tag{2.3}$$

Following Siegman [9] and Talman [10] we want to interpret this integral as the convolution

$$(H_\nu f)(e^x) = F \otimes j_\nu(x), \tag{2.4}$$

where

$$F(x) = e^{-x} f(e^{-x}), \qquad j_\nu(x) = J_\nu(e^x). \tag{2.5}$$

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where

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This interpretation requires $F, j_\nu \in L^2(\mathbb{R})$, $L^2(\mathbb{R})$ the space of square-integrable functions on the real line. The function F certainly belongs to $L^2(\mathbb{R})$ if f is bounded, continuous, and if $|f(\lambda)| \sim \mathcal{O}(\lambda^{-p})$, $p>1$, for $\lambda \to \infty$. The function $j_\nu(x)$ decays exponentially for $x \to \infty$, and it decays exponentially for $x \to -\infty$, provided $\nu > 0$. Hence, $j_\nu \in L^2(\mathbb{R})$ for $\nu > 0$.

The convolution may be computed by means of the convolution theorem $\mathcal{F}(F \otimes j_\nu) = \mathcal{F}F \times \mathcal{F}j_\nu$, with $\mathcal{F}$ the Fourier transform given by

$$\hat{f}(s) \underset{\mathrm{D}}{=} \mathcal{F}f(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} f(t)\, dt. \tag{2.6}$$

Making use of formula (11.4.16) in Abramowitz and Stegun [1], it is possible to compute the Fourier transform of j_ν explicitly, cf. Talman [10]. We have

$$\hat{j}_\nu(s) = \frac{1}{\sqrt{2\pi}} 2^{-is-1} \left(\Gamma\left(\frac{\nu}{2} - \frac{is}{2}\right) \Big/ \Gamma\left(\frac{\nu}{2} + \frac{is}{2} + 1\right)\right). \tag{2.7}$$

Apply the formula $\Gamma(z+1) = z\Gamma(z)$ to the denominator in (2.7), and recall that $|\Gamma(z)| = |\Gamma(\bar{z})|$. Then

$$\hat{j}_\nu(s) = \frac{1}{\sqrt{2\pi}} 2^{-is} \frac{1}{\nu + is} e^{2i \arg(\Gamma(\nu/2 - is/2))}. \tag{2.8}$$

Formula (2.8) shows that the Fourier transform $\hat{j}_\nu$ of $j_\nu(x) = J_\nu(e^x)$ decays like $1/|s|$ for large $|s|$.

We return to the convolution (2.4). The Fourier transform of the Bessel function term in the convolution decays slowly. However, if the other term, the function $\hat{F}$, decays rapidly, then the product of the two transforms decays rapidly. I.e., the Fourier transform of $H_\nu f$ then decays rapidly. In such circumstances the approximation of the Fourier transform of F by a discrete Fourier transform (DFT) on an interval of suitable length is feasible. This is exactly the algorithm, see also Talman [10].

ALGORITHM 2.1 (Approximation of the Hankel transform H_ν, $\nu > 0$). (i) Choose an appropriate value for T and approximate F, cf. (2.5), on $(-T, T]$ by means of a trigonometric polynomial. T should be so large that the difference between the trigonometric polynomial and F is sufficiently small on $[-T, T]$. Use the FFT in the computation of the Fourier coefficients.

(ii) Compute the corresponding values of $\hat{j}_\nu$ by means of formula (2.8). The argument of the gamma function may be computed using the Kuki algorithm, cf. [7, Algorithm 421].

(iii) Apply the convolution theorem; i.e., compute the product of corresponding Fourier coefficients of $\mathcal{F}F$ and $\hat{j}_\nu$ and obtain an approximating trigonometric polynomial on $(-T, T]$ for $(H_\nu f)(e^x)$ by means of the FFT.

This algorithm is essentially the one described by Talman [10] for the Fourier–Bessel transform. The gamma function is evaluated using the Kuki algorithm, see also Kolbig [6]. Talman [10] describes a similar process. For the Hankel transform Talman [10] prefers the formula

$$(H_\nu f)(e^x) = e^{-\alpha x} \int_{-\infty}^{\infty} e^{\alpha y} e^{-y} f(e^{-y})\, e^{\alpha(x-y)} \times J_\nu(e^{x-y})\, dy \tag{2.9}$$

instead of (2.3). However, in our computations we found $\alpha = 0$ the best value. The reason is quite simple. The integral in (2.9) is computed as a convolution by means of discrete (fast) Fourier transforms. Then, either the results for small x or the results for large x are multiplied by an exponentially large factor. We found that this multiplication by $\exp(-\alpha x)$ spoils the results. For $\alpha = 0$ there is no multiplication by a large factor.

By using (2.9) instead of (2.3) Talman [10] is able to apply his algorithm to approximate H_0, but with $\alpha \neq 0$. In the following section we shall describe a slightly modified algorithm for the approximation of $H_0 f$ without a multiplication by an exponential factor.

3. THE HANKEL TRANSFORMATION H_0

In this section we consider the Hankel transformation H_0. We obtain a useful algorithm in which the FFT is interpreted by means of principal value integrals. Consider formula (2.3) with $\nu = 0$. The formula is valid for all F, cf. (2.5) of class $L^2(\mathbb{R})$. However, it is not possible to interpret the integral (2.3) as a convolution, since the function j_0 is not of class $L^2(\mathbb{R})$. In order to obtain a convolution interpretation we rewrite (2.3) with $\nu = 0$ as, cf. Talman [10], or (2.9):

$$(H_0 f)(e^x) = e^{-\alpha x} \int_{-\infty}^{\infty} e^{-(1-\alpha)y} f(e^{-y})\, e^{\alpha(x-y)} \times J_0(e^{x-y})\, dy. \tag{3.1}$$

Put

$$\begin{aligned} F_\alpha(x) &= e^{-(1-\alpha)x} f(e^{-x}), \\ j_{0,\alpha}(x) &= e^{\alpha x} J_0(e^x). \end{aligned} \tag{3.2}$$

We choose $\alpha > 0$, and we suppose $F_\alpha \in L^2(\mathbb{R})$ for sufficiently small non-negative α. Later in this section we shall summarize the requirements on the functions F_α and f. Note that $F_0 = F$, cf. (2.5). Clearly, for small positive α we have $j_{0,\alpha} \in L^2(\mathbb{R})$. Hence, the integral in (3.2) may be interpreted as a convolution, provided $\alpha > 0$. By the convolution theorem we have

$$(H_0 f)(e^x) = e^{-\alpha x} \mathcal{F}^{-1} \hat{F}_\alpha \times \hat{j}_{0,\alpha}(x) \tag{3.3}$$

In view of formula (11.4.16) of Abramowitz and Stegun [1] we easily compute $j_{0,\alpha}$; see also Talman [10],

$$j_{0,\alpha} = \frac{1}{\sqrt{2\pi}} 2^{\alpha - is} \frac{\Gamma(1 + \frac{1}{2}\alpha - \frac{1}{2}is)}{\Gamma(1 - \frac{1}{2}\alpha + \frac{1}{2}is)} \frac{1}{\alpha - is}.$$

We want to take the limit $\alpha \to 0$ in formula (3.3). Clearly, this limit exists provided

$$\lim_{\alpha \to 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{isx} \hat{F}_\alpha(s)\, 2^{-is} \frac{\Gamma(1 - \frac{1}{2}is)}{\Gamma(1 + \frac{1}{2}is)} \frac{1}{\alpha - is}\, ds \tag{3.4}$$

exists. Assume for small positive α in $L^2(\mathbb{R})$-sense,

$$\hat{F}_\alpha = \hat{F} + \alpha G + \mathcal{O}(\alpha^2). \tag{3.5}$$

This formula holds if F_α, $tF_\alpha(t)$, and $t^2F_\alpha(t)$ belong to $L^2(\mathbb{R})$, and this is so if f is bounded and if $|f(t)| = \mathcal{O}(t^{3+\varepsilon})$, $\varepsilon > 0$. These conditions also ensure the differentiability of $\hat{F}_\alpha$. Using the differentiability of $\hat{F}$ and (3.5) we see that the limit in (3.4) exists and is given by the *principal value* integral

$$\int_{-\infty}^{\infty} e^{isx} \hat{F}(s)\, 2^{-is} \frac{\Gamma(1 - \frac{1}{2}is)}{\Gamma(1 + \frac{1}{2}is)} \frac{1}{-is}\, ds.$$

Hence under rather weak assumptions on f,

$$(H_0 f)(e^x) = -\frac{1}{2\pi} \text{P.V.} \int_{-\infty}^{\infty} e^{isx} \hat{F}(s)\, 2^{-is} \cdot \frac{\Gamma(1 - \frac{1}{2}is)}{\Gamma(1 + \frac{1}{2}is)} \frac{1}{is}\, ds. \tag{3.6}$$

It should be noted that this formula may also be obtained from the Parseval relation for the Mellin transform, cf. Wong [13, p. 152].

Let us now consider the approximation of the principle value integral in (3.6). Again as in Section 2 we want to use the DFT (implemented as a FFT) to obtain the value of the right-hand side in equidistant points on the x-axis. As in Section 2 we restrict the domain of integration on the x-axis to $(-T, T]$ and in the frequency domain to $(-S, S]$. Here S depends on T and the number of points in the DFT. The DFT uses values of the function

$$\hat{K}(s) = \frac{\hat{k}(s)}{is}, \qquad \hat{k}(s) = \hat{F}(s)\, 2^{-is} \frac{\Gamma(1 - \frac{1}{2}is)}{\Gamma(1 + \frac{1}{2}is)} \tag{3.7}$$

in symmetrically (with respect to $s = 0$) placed equidistant points on the s-axis, including $s = 0$. These values are just the Fourier coefficients of the trigonometric polynomial associated with the DFT. Typically, the value of $\hat{K}$ at $s = 0$ is undefined. We remedy this by putting $\hat{K}(0) = 0$. One obtains $\hat{K}(0) = 0$ by modifying $\hat{k}$ in a small neighborhood of $s = 0$, such that

$$\lim_{s \to 0} \frac{\hat{k}(s)}{s} = 0.$$

The diameter of the neighborhood should also be so small that the other values of $\hat{K}$ used in the DFT are not affected. This is always possible. The DFT on $(-S, S]$ is now well defined, but the result of the computation may suffer from the deliberate choice $\hat{K}(0) = 0$. Let us consider the consequences of this choice. The key observation is the factor is in the denominator of $\hat{K}$. In view of this factor we see that $\mathcal{F}^{-1}\hat{k}$ is the derivative of $\mathcal{F}^{-1}\hat{K}$. This is formally true for the Fourier transform. However, it is always true for the DFT on $(-S, S]$. Let us denote the DFT operator by $\mathcal{F}_S$. Hence, with the interpretation as trigonometric polynomial,

$$\frac{d}{dx} \mathcal{F}_S^{-1} \hat{K} = \mathcal{F}_S^{-1} \hat{k},$$

or

$$\mathcal{F}_S^{-1} \hat{K} = \int \mathcal{F}_S^{-1} \hat{k}\, dx.$$

The deliberate choice of $\hat{K}(0) = 0$ may be explained by a change of $\hat{k}$ on a small neighborhood of the origin, see above. In this neighborhood of the origin the DFT uses only the constant function component. Thus the modification of $\hat{k}$ implies a modification by a constant function of $\mathcal{F}_S^{-1}\hat{k}$. Therefore, with the choice $\hat{K}(0) = 0$, the primitive function $\int \mathcal{F}_S^{-1}\hat{k}\, dx$ is determined modulo a linear function in x. This means that the DFT approximation of (3.6) in combination with $\hat{K}(0) = 0$ determines an approximation of $(H_0 f)(e^x)$ modulo a linear function in x. Fortunately, this suffices, since $(H_0 f)(0)$ and $(H_0 f)(\infty)$ are easily obtained. Indeed,

$$(H_0 f)(\infty) = 0, \qquad (H_0 f)(0) = \int_{-\infty}^{\infty} e^{-y} f(e^{-y})\, dy = \hat{F}(0). \tag{3.8}$$

This results in the following algorithm.

ALGORITHM 3.1 (Approximation of the Hankel transform H_0). (i) Choose an appropriate value for T and approximate F, cf. (2.5) on $(-T, T]$ by means of a trigonometric polynomial. T should be so large that the difference between the trigonometric polynomial and F is sufficiently small on $[-T, T]$. Use the FFT in the computation of the Fourier coefficients.

(ii) Compute the corresponding values of $\hat{K}$, cf. (3.7), but put $\hat{K}(0)=0$. Use the Kuki algorithm, cf. [7, Algorithm 421], for the computation of the argument of the Γ-function.

(iii) Compute $\mathscr{F}_S^{-1}\hat{K}$ via the FFT, cf. (3.7) for $\hat{K}$.

(iv) Adjust the values of $\mathscr{F}_S^{-1}\hat{K}$ by a linear function in x in such a way that the approximation in $x=-T$, i.e., $r=e^{-T}$ equals $\hat{F}(0)$, and such that the approximation in $x=T$, i.e., $r=e^{T}$ equals 0. This adjusted function in the variable x approximates $(H_0 f)(e^x)$.

4. NUMERICAL ILLUSTRATION

We illustrate the performance of Algorithm 3.1 by means of two simple examples. In the first example we choose

$$f(\lambda)=e^{-3\lambda/2}-e^{-5\lambda/2} \tag{4.1a}$$

and in the second example we choose

$$f(\lambda)=\lambda\exp(-\tfrac{1}{2}\lambda^2) \tag{4.1b}$$

In both instances the transform $H_0 f$ is known explicitly, and it is given by

$$(H_0 f)(r)=\frac{1}{\sqrt{r^2+9/4}}-\frac{1}{\sqrt{r^2+25/4}} \tag{4.2a}$$

and

$$(H_0 f)(r)=\exp(-\tfrac{1}{2}r^2), \tag{4.2b}$$

respectively. We approximate the transformed function by the original Siegman–Talman algorithm. I.e., we implement the algorithm based on (3.1). Since $j_{0,\alpha}\in L^2(\mathbb{R})$ we must have $\alpha>0$. In Table I we list the maximum error in modulus at the sampling points. In the last row of Table I we give the error for Algorithm 3.1. In these examples we have

$$n=256, \qquad T=30.$$

By choosing this rather large value of T we can be sure that the discrete Fourier transform is a good approximation of the Fourier transform if n is large enough.

It is clear that the error in the approximation of Algorithm 3.1 is smaller than the error in the original Siegman–Talman algorithm. For small values of α the factor $\alpha-is$ in the denominator in (3.4) spoils the results, while for larger α the exponential factor $\exp(\alpha T)$ spoils the results in a neighborhood of $r=0$. For larger α the errors in Table I are attained at $r=\exp(-T)$. These results exhibit the bad effects of premultiplication by an exponential factor as required by (3.1). Also, the improvement offered by Algorithm 3.1 is obvious.

TABLE I

α	Problem (4.1a)	Problem (4.1b)
0.05	1.397E−02	5.240E−02
0.10	6.626E−04	2.490E−03
0.15	3.291E−05	1.290E−04
0.20	1.639E−06	1.185E−05
0.25	8.193E−08	1.929E−05
0.30	5.628E−09	7.944E−05
0.35	7.129E−09	3.701E−04
0.40	3.070E−08	1.931E−03
0.45	1.363E−07	1.008E−02
0.50	6.086E−07	5.221E−02
0.55	2.732E−06	2.686E−01
0.60	1.233E−05	1.374E+00
0.65	5.605E−05	6.995E+00
0.70	2.567E−04	3.545E+01
0.75	1.185E−03	1.789E+02
Algorithm 3.1	5.885E−11	5.383E−06

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